Growth

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Exercise 1. Population Growth in the Solow Model

Consider the Solow model in discrete time. The production function is Cobb–Douglas:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}, \qquad 0 < \alpha < 1,$$

where $A > 0$ is constant. Capital accumulates according to

$$K_{t+1} = (1-\delta)K_t + sY_t, \qquad 0 < \delta < 1,\ 0 < s < 1.$$

Population grows at the exogenous rate $n \geq 0$:

$$L_{t+1} = (1+n)L_t.$$

(a) Define capital per worker $k_t \equiv K_t/L_t$ and output per worker $y_t \equiv Y_t/L_t$. Show that

$$y_t = Ak_t^{\alpha}.$$

(b) Derive the law of motion for capital per worker:

$$k_{t+1} = \frac{(1-\delta)k_t + sAk_t^{\alpha}}{1+n}.$$

Then show that the change in capital per worker can be written as

$$\Delta k_{t+1} = k_{t+1} - k_t = \frac{sAk_t^{\alpha} - (\delta+n)k_t}{1+n}.$$

Explain why population growth acts like an additional depreciation term.

(c) A steady state satisfies $k_{t+1} = k_t = k^*$.

Derive the steady-state level of capital.

How does an increase in n affect:

- steady-state capital per worker k^*?
- steady-state output per worker y^*?
- steady-state consumption per worker?

(d) In steady state, what is the growth rate of:

- total output Y_t?
- output per worker y_t?

(e) **Demographic Reform (One-Child Policy Scenario).**

Suppose the economy is initially in steady state with population growth rate $n > 0$.

At time $t = 0$, the government permanently reduces population growth to $n' < n$.

- Using the steady-state formula, compare the old steady state k^* with the new steady state k^{**}.
- Immediately after the reform, is $k_1 - k_0$ positive, negative, or zero?
- Describe qualitatively the transition dynamics of:
  - capital per worker,
  - output per worker,
  - total output.
- Compare the long-run growth rate of total GDP before and after the reform.

Exercise 2. AI Shock in Romer Model

Consider the Romer model of endogenous growth, where the production of new ideas depends on the productivity of research δ_A:

$$A_{t+1} - A_t = \delta_A A_t L_A$$

where A_t is the stock of ideas and L_A is the number of researchers. Output per person is given by $y_t = A_t(1 - \ell)$, where ℓ is the fraction of labor allocated to research.

Suppose a breakthrough in artificial intelligence (AI) increases the productivity of research, raising δ_A.

(a) Illustrate the effect of the AI shock on the time path of $\log y_t$ (a sketch or description is sufficient).

(b) Explain the economic intuition: How does a higher δ_A affect long-run growth? What happens to the level and growth rate of output per person?

(c) How would your answer change if the AI shock also allowed a larger share of labor to be allocated to research (ℓ increases)?