Chapter II: A Model of Production

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What Is Happening to the Economy?

“AI Reshapes Work — Payrolls Barely Rise Despite Record Productivity”

Fortune

“Europe’s Economy Continues to Lag Behind the U.S.”

Financial Times

“Chega Defends ‘Portugueses Primeiro’ in Economic Policy”

Público

Two Ways to Answer Big Economic Questions

- There are two main approaches in economics:
- **1. Empirical approach**
  - Use data to estimate causal effects.
  - Look for natural experiments.
  - Compare treated vs. control groups.
- **2. Theoretical approach**
  - Build a model of the whole economy.
  - Make assumptions explicit.
  - Study how all markets adjust simultaneously.

Why Isn't Empirical Evidence Enough?

- First problem: **No natural experiment for everything**
  - We cannot randomly assign an economy to "AI revolution".
- Second problem: equilibrium effects
  - A policy affects one group...
  - But prices adjust for everyone
  - Other markets respond.
  - No control group
- Example:
  - Education: Should a government subsidize university education?
- In these case we need a model
- A model is said to be in **general equilibrium** when both prices and quantities are determined within the model.

Production Function

- If we want to understand:
  - How AI affects wages
  - Why the U.S. produces more than Europe
  - What happens when the labor force increases
- We must first answer a simpler question: **How is total output produced?**
- Economists summarize this with a production function:

$$Y = F(K, L)$$

- Output depends on:
  - L — workers (labor)
  - K — machines
- This simple relationship will not only determine output determine wages and profits.

What Happens When We Add More Workers?

- Hold capital fixed (K constant).
- Suppose the labor force increases (migration) or decreases (aging).
- We assume that adding more workers has two effects:
  - Total output rises — more hands help produce more.
  - But each additional worker adds a bit less than the previous one — they share the same capital and tools.
- Mathematically, we capture this with:
  - Positive marginal product: $\frac{\partial F}{\partial L} > 0$
  - Diminishing marginal product: $\frac{\partial^2 F}{\partial L^2} < 0$
- These assumptions make our production function realistic and will allow us to reason about labor, wages, and migration.

What Happens When We Add More Machines?

- Hold labor fixed (L constant).
- Suppose the firm invests in more machines or equipment.

- We assume that adding more capital has two effects:
  - Total output rises — more machines help produce more.
  - But each additional machine adds a bit less than the previous one — workers have only so much to operate.

- Mathematically, we capture this with:
  - Positive marginal product: $\frac{\partial F}{\partial K} > 0$
  - Diminishing marginal product: $\frac{\partial^2 F}{\partial K^2} < 0$

- These assumptions help us reason about why output per capita differs across countries

Cobb–Douglas Production Function

- A very popular production function in macro is the Cobb–Douglas:

$$Y = AK^{a}L^{1-a}$$

- A measures productivity: how good the economy is at turning inputs into output.
- The marginal products are easy to compute:

$$\frac{\partial Y}{\partial L} = (1-a)AK^{a}L^{-a} \geq 0, \qquad \frac{\partial Y}{\partial K} = aAK^{a-1}L^{1-a} \geq 0$$

- And both marginal products get smaller as we increase that input:

$$\frac{\partial^2 Y}{\partial L^2} \leq 0, \qquad \frac{\partial^2 Y}{\partial K^2} \leq 0$$

- The two inputs also help each other:

$$\frac{\partial^2 Y}{\partial K \partial L} = a(1-a)AK^{a-1}L^{-a} \geq 0$$

Adding machinery raises how much extra output you get from hiring an extra

Returns to Scale: What Happens if We Grow Everything?

- Recall our Cobb–Douglas technology:

$$Y = AK^{a}L^{1-a}$$

- What happens if we double both capital and labor?

$$F(2K, 2L) = A(2K)^{a}(2L)^{1-a} = 2\,F(K, L)$$

- Doubling inputs gives exactly double output — this is called **constant returns to scale**.

Why Constant Returns to Scale?

- For Cobb–Douglas, returns to scale depend on the sum of the exponents:

$$a + (1 - a) = 1$$

- If the sum is:
  - Equal to 1: constant returns to scale
  - Greater than 1: increasing returns
  - Less than 1: decreasing returns

Firm's Problem

- So far, we have seen how output responds to changes in labor and capital.
- Now, let's see how a firm decides how many workers and machines to hire in practice.
- Firms choose K and L to maximize profits:

$$\max_{K,L} \; F(K,L) - rK - wL$$

- Profits = production minus cost of inputs.
- We set the price of output to 1 (numeraire).
- r = rental price of capital, w = wage of workers.
- Firms take r and w as given because markets are competitive (simplifying assumption — left-wing critique possible).

Firm's Problem: Solution

- Profit-maximization problem:

$$\max_{K,L} \Pi(K,L) = \max_{K,L} \; F(K,L) - rK - wL$$

- Step 1: Choose optimal capital K (hold L fixed)
- Step 2: Choose optimal labor L (hold K fixed)

- We will see that the solution satisfies:

$$\mathrm{MP}_K(K^*,L^*) = \frac{\partial F(K^*,L^*)}{\partial K} = r, \quad \mathrm{MP}_L(K^*,L^*) = \frac{\partial F(K^*,L^*)}{\partial L} = w$$

  Just like in micro: hire/rent until marginal revenue = marginal cost.

- MP_K and MP_L are the marginal product of capital and labor.

Maximization in Two Variables

- General math rule:

$$\max_{x,y} f(x, y)$$

- Solution satisfies first-order conditions:

$$\frac{\partial f(x^*, y^*)}{\partial x} = 0, \quad \frac{\partial f(x^*, y^*)}{\partial y} = 0$$

- Applied to the firm:

$$\frac{\partial \Pi(K^*, L^*)}{\partial K} = 0, \quad \frac{\partial \Pi(K^*, L^*)}{\partial L} = 0$$

- Intuition: hire each input until marginal revenue of the input = cost of the input (just like MR = MC in micro for one good).

Firm's Problem with Cobb–Douglas

- Profit function:

$$\Pi(K, L) = F(K, L) - rK - wL$$

- Cobb–Douglas:

$$\Pi(K, L) = AK^a L^{1-a} - rK - wL$$

- Goal: choose K and L to maximize profits.
- Taking the derivative and equating to zero we get:

$$aAK^{a-1}L^{1-a} - r = 0, \quad (1-a)AK^a L^{-a} - w = 0$$

$$\mathrm{MP}_K = aAK^{a-1}L^{1-a} = r, \quad \mathrm{MP}_L = (1-a)AK^a L^{-a} = w$$

- Hire inputs until MP = price (marginal revenue = marginal cost).

Firm's Problem

Example: Hiring Labor with Fixed Capital

- Imagine that you own 1 machine and decide how many workers to hire.
- Profit function:

$$\Pi(K, L) = AK^{a}L^{1-a} - wL$$

- FOC for labor:

$$\frac{\partial \Pi}{\partial L} = A(1-a)K^{a}L^{-a} - w = 0$$

- Numbers: $A = 1, a = 0.5, K = 1, w = 0.25$

$$0.5 \cdot 1^{0.5}L^{-0.5} - 0.25 = 0 \implies L = 4$$

- Intuition: Hire workers until marginal product of labor = wage.

Firm Behavior: Marginal Product of Labor

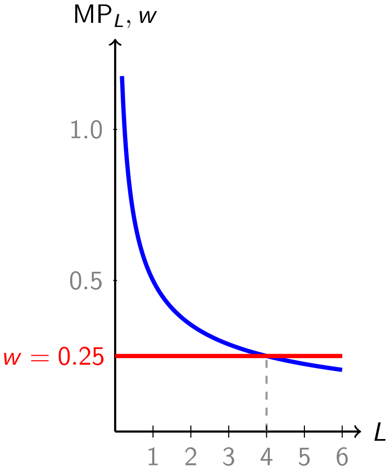
- Marginal product of labor:

$$\text{MP}_L = (1-a)AK^aL^{-a}$$

- Downward sloping: each additional worker adds less output.
- Horizontal line = wage w.
- Hire labor until:

$$\text{MP}_L = w$$

- Intersection shows optimal number of workers L^*.



Hiring Capital with Fixed Labor

- Imagine that you are a firm owner with 1 worker and want to decide how many machines to rent.
- Profit function:

$$\Pi(K, L) = AK^{a}L^{1-a} - rK - wL$$

- Numbers: $A = 1, a = 0.5, L = 1, r = 1, w = 2$
- First-order condition for capital:

$$\frac{\partial \Pi}{\partial K} = aK^{a-1}L^{1-a} - r = 0$$

- Plug in numbers:

$$0.5K^{-0.5} \cdot 1^{0.5} - 1 = 0 \implies K = 0.25$$

- Rent machines until marginal revenue = marginal cost: rental price.

Firm's Problem

Example: Both Capital and Labor

- Story: Now the firm chooses both K and L. Prices: $r = 1$, $w = 0.25$. Technology: $A = 1$, $a = 0.5$.
- Profit-maximization first-order conditions:

$$\frac{\partial \Pi}{\partial K} = aK^{a-1}L^{1-a} - r = 0, \quad \frac{\partial \Pi}{\partial L} = (1-a)K^{a}L^{-a} - w = 0$$

- Numbers give ratios:

$$\frac{MP_L}{MP_K} = \frac{(1-a)K^{a}L^{-a}}{aK^{a-1}L^{1-a}} = \frac{1-a}{a}\frac{K}{L} = \frac{w}{r} \implies \frac{K}{L} = \frac{aw}{(1-a)r} = \frac{0.5 \cdot 0.25}{0.5 \cdot 1} = 0.25$$

- The firm would scale infinitely with constant returns to scale, but the capital-to-labor ratio is well-defined from the MRTS = input price ratio you saw in microeconomics.

Capital and Labor Supply

- We have seen how firms choose capital and labor to maximize profits
- Firms' decisions determine factor demand
- Now we introduce factor supply.
- Simplifying assumption:

$$K^s = \overline{K}, \qquad L^s = \overline{L}$$

(Fixed total amounts in the economy.)

Model Summary

- The model consists of five equations:
- Capital demand: $aAK^{a-1}L^{1-a} = r$
- Labor demand: $(1-a)AK^{a}L^{-a} = w$
- Capital supply: $K = \overline{K}$
- Labor supply: $L = \overline{L}$
- Production function: $Y = AK^{a}L^{1-a}$
- The five endogenous variables are: K, L, r, w, and Y.

An Equilibrium

- In economics, the solution to a model is called an equilibrium.
- Equilibrium describes what happens when all markets clear, that is, supply equals demand in each market.
- In our model, we have five equations with five unknowns.
- Solving the model means finding the values of the endogenous variables (K, L, r, w, Y) in terms of the exogenous variables ($\overline{K}$, $\overline{L}$, A, a).
- This involves rewriting the system so that all endogenous variables are on one side of the equations and only parameters and exogenous variables on the other side.

Equilibrium Values

- In our simple model, the equilibrium values are:
- Capital: $K = \overline{K}$
- Labor: $L = \overline{L}$
- Rental rate of capital: $r = aA\overline{K}^{a-1}\overline{L}^{1-a}$
- Wage: $w = (1-a)A\overline{K}^{a}\overline{L}^{-a}$
- Output: $Y = A\overline{K}^{a}\overline{L}^{1-a}$
- These values satisfy all five equations of the model and clear labor, capital and goods markets.

Labor Share in Our Model

- Labor compensation is wL. The labor share of output is:

$$\text{Labor share} = \frac{wL}{Y}$$

- From the labor demand curve:

$$w = (1-a)AK^{a}L^{-a} = (1-a)\frac{Y}{L}$$

- Multiply both sides by L to get total labor compensation:

$$wL = (1-a)Y$$

- Therefore, labor receives a constant fraction $1-a$ of total output in this model.

Why Cobb-Douglas

- Cobb-Douglas is easy to work with mathematically.
- More importantly, it implies that the share of output going to each factor is constant over time.
- Empirically, labor and capital shares have been surprisingly stable since World War II: $\alpha = 1/3$

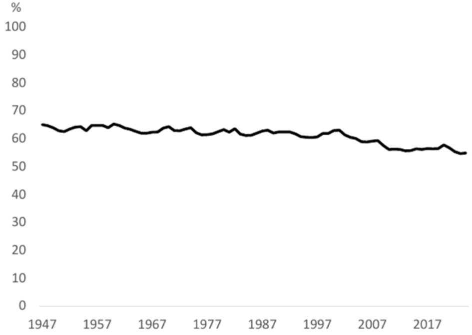


Figure 2: Labor Share in U.S. Nonfarm Business Sector

Development Accounting

- We now make an important leap: we apply the production function of our model to aggregate economies across the world.
- Output per person in the model, y, is matched to **GDP per capita**.
- Capital, k, is measured as the economy's stock of housing, factories, machines, and equipment, divided by population.
- The production function implies:

$$y = Ak^{1/3}.$$

- For now, we make a strong simplifying assumption:

$$A = 1 \quad \text{for all countries},$$

  so that income differences depend *only* on capital per person:

$$y = k^{1/3}.$$

- To ease comparisons, we normalize all variables so that

Predictions of the Model

Consider Portugal as an example.

- Capital per person in Portugal and Spain is close to or even slightly above the U.S. level.

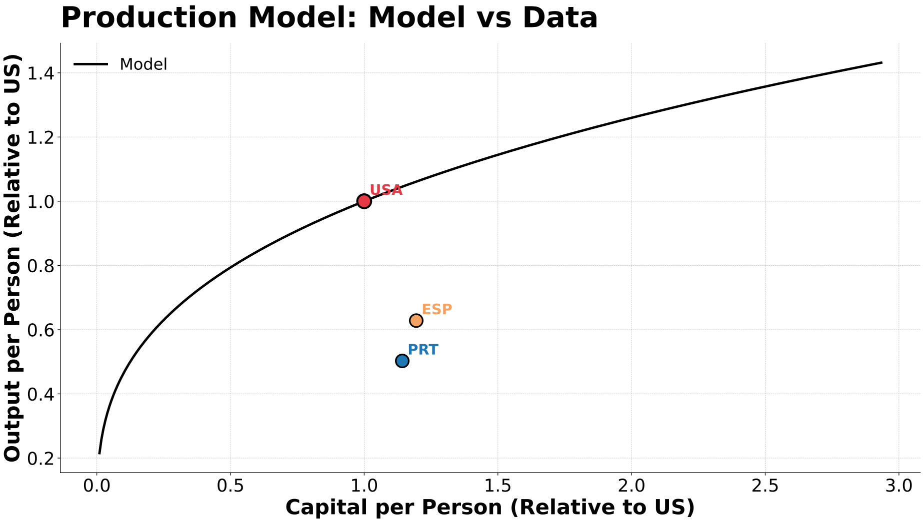
According to the model:

$$y = k^{1/3},$$

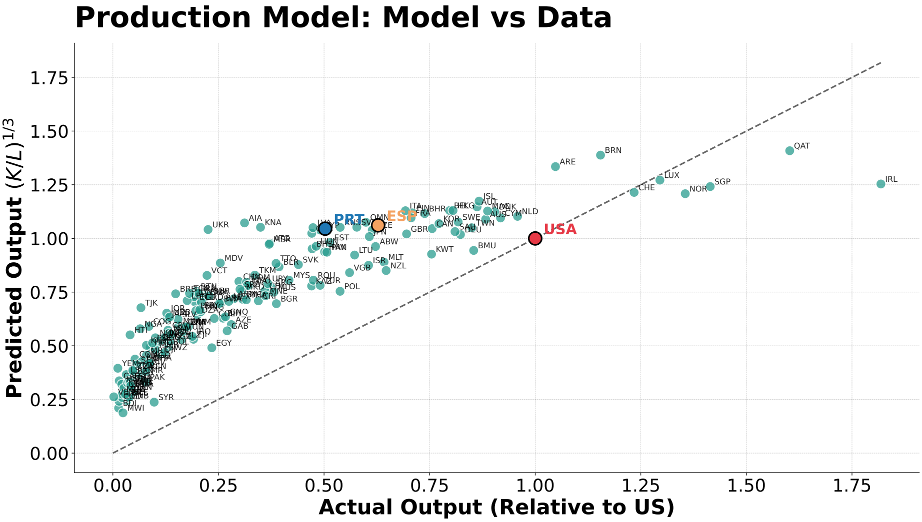
so if $k_{\text{PRT}} \approx 1$, the model predicts:

$$y_{\text{PRT}} \approx 1.$$

Prediction: Portugal should be about as rich as the United States.



Source: Penn World Table 11.0 (Feenstra, Inklaar, and Timmer). Year: 2023.

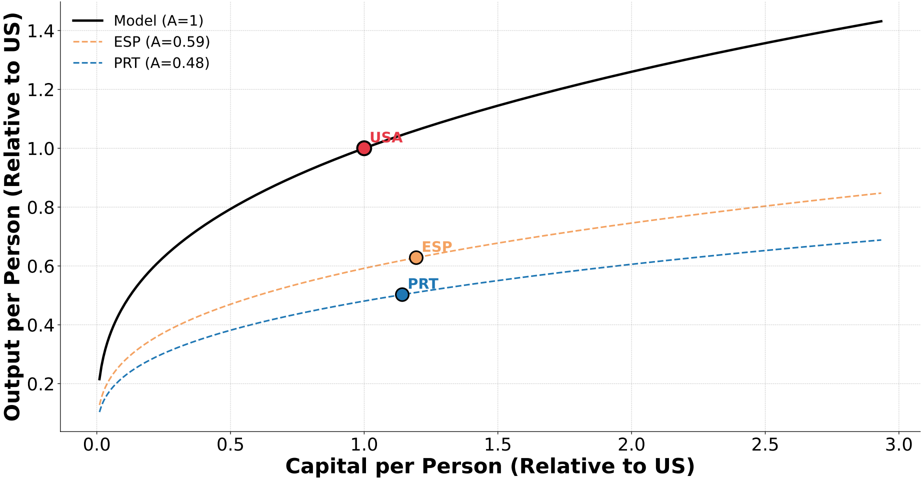


Source: Penn World Table 11.0 (Feenstra, Inklaar, and Timmer). Year: 2023.

Model vs Data

- In general, the model correctly predicts that countries are rich or poor according to how much capital per person they have.
- However, the magnitudes are very wrong.
- One way to reconcile the model with the data is to allow the productivity parameter A to be country-specific.
- A measures how productive countries are at using their factor inputs (in this case, K and L) to produce output.
- Economists typically assume the model is correct and use it to infer the value of A. For example: what value of A would rationalize why Portugal produces much less output per capita than the U.S.?
- The inferred value of A is referred to as total factor productivity (TFP).

Production Model with Country-Specific TFP



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Model vs Data

- Poor countries have 70 times lower GDP per capita than the US
- If poor countries had the US TFP they would be around 14 times richer.
- What drives differences in TFP?
  - Education
  - Technoloy
  - Institutions
  - Misallocation

Example: Cabo Verde

- Cabo Verde has approximately 20% of the capital per capita of the US:

$$k_{\text{CPV}} = 0.20$$

- The model predicts output per capita:

$$y_{\text{CPV}} = k_{\text{CPV}}^{1/3} = 0.20^{1/3} = 0.585$$

  So the model predicts Cabo Verde should be about **60%** as rich as the US.

- In the data, Cabo Verde is about **15%** as rich as the US:

$$y_{\text{CPV}}^{\text{data}} = 0.15$$

- The model overpredicts gdp per capita by a large margin — capital alone cannot explain the gap.

Example: Cabo Verde

- We infer TFP by solving $y = Ak^{1/3}$:

$$A_{\text{CPV}} = \frac{y_{\text{CPV}}^{\text{data}}}{k_{\text{CPV}}^{1/3}} = \frac{0.15}{0.585} = 0.26$$

- Decomposing the income gap:

$$\underbrace{\frac{y_{\text{CPV}}}{y_{\text{US}}}}_{0.15} = \underbrace{\left(\frac{k_{\text{CPV}}}{k_{\text{US}}}\right)^{1/3}}_{0.585 \text{ (capital)}} \times \underbrace{A_{\text{CPV}}}_{0.26 \text{ (TFP)}}$$

- The US is 6.6 (=1/0.15) times richer than Cabo Verde.
  - If it had the same TFP as the US, it would be 3.85 (=1/0.26) times richer.
  - If it had the same capital as the US, it would be 1.71 (=1/0.585) times richer.
- Most of Cabo Verde's gdp gap relative to the US is due to lower productivity.

Worries about Labor-Saving Technology

- People have worried since the Industrial Revolution that machines will take jobs.
- Despite large amounts of labor-saving technology, labor share has remained roughly constant.
- This is very different from what happened with horses, which were replaced by machines and effectively disappeared from production.
- The question today: will new technologies like AI and robots change labor share in the future?

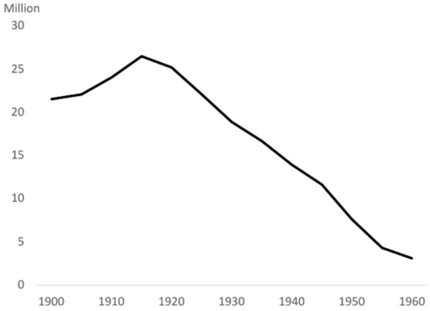

Figure 3: Number of Horses and Mules in the United States, 1900-1960

- One reason labor share might fall is better or cheaper machines.
- In a Cobb-Douglas production function:

$$Y = A(zK)^a L^{1-a}$$

where z measures machine quality.

- The rental rate of capital is:

$$r = aAK^{a-1}L^{1-a}z^a = a\frac{Y}{K}$$

- Total payments to capital:

$$rK = aY$$

- Labor share remains $1 - a$, independent of z. Cobb-Douglas cannot capture a falling labor share due to better machines.

Is AI Going to Steal Our Jobs?

- One reason labor share might fall is better or cheaper machines.
- In a Cobb-Douglas production function:

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- Labor share remains $1 - a$, independent of z. Cobb-Douglas cannot capture a falling labor share due to better machines.
- With a different production function (e.g., CES with elasticity > 1), labor share falls if machines improve or become cheaper.