Equilibrium with Complete Markets

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Introduction

- This course is an introduction to modern macroeconomic theory.
- Our main emphasis will be the analysis of resource allocations in dynamic stochastic environments.
- We will go though the analysis of:
  - Equilibrium with complete markets.
  - Dynamic Programming (DP)
  - Applications of DP (RBC models)
- We will start, however, with a simple environment: static exchange economy.

References: *Recursive Macroeconomic Theory* by Ljungqvist and Sargent and *The PhD Macro Book*

Exchange Economy

- Simple environment: finite dimensional, static exchange economy.
- In an exchange economy, people interact in the market place.
- They buy and sell goods taking market prices as given in order to maximize their utility.
- Their choices are constrained by their endowments.

Exchange Economy

- If we can find a set of selling and buying decision for all individuals and a set of prices such that:
  - Given these prices, people's selling and buying decision are optimal.
  - No excess demand or excess supply of any good.

$\Rightarrow$ Our economy is in equilibrium.

Setup

- Consider an economy with $i = 1, \ldots, n$ consumers and $j = 1, \ldots, m$ commodities.
- Each individual i is endowed with w_i^j units of good j. $(w_i^1, w_i^2, \ldots, w_i^m)$
- Individuals have preferences over these goods and will trade with each other to maximize their well-being.

Assumptions

1. Consumer's preferences are representable by a utility function $u.u_i : \boldsymbol{X} \equiv \mathbb{R}_+^m \rightarrow \mathbb{R}$
2. u is continuous and first and second derivatives exist.
3. Preferences are strictly monotonic (the more I consume, the better).
4. u is strictly concave (no flat section in indifference curves).
5. Every agent is endowed with a positive amount of each good.
6. $\| Du_i(x_k) \| \rightarrow \infty$ as $x_k \rightarrow x$ where some component of x is equal to zero.

Problem

- Given a set of prices $\boldsymbol{p} = (p^1, \dots, p^m)'$, consumers in this economy solve the following problem:

$$\max_{\boldsymbol{x}_i} u_i(\boldsymbol{x}_i)$$
$$\text{s.t.} \quad \boldsymbol{p}'(\boldsymbol{x}_i - \boldsymbol{w}_i) \leq 0$$

  Given that preferences are monotonic, individuals will be on their budget set: $\boldsymbol{p}'(\boldsymbol{x}_i - \boldsymbol{w}_i) = 0$

- Following is the Lagrangian of the consumer problem:

$$\mathcal{L} = u_i(\boldsymbol{x}_i) - \mu_i \boldsymbol{p}'(\boldsymbol{x}_i - \boldsymbol{w}_i)$$

Problem
FOCs

- FOCs are necessary and sufficient to characterize $\boldsymbol{x}_i$:

$$D_x u_i(\boldsymbol{x}_i) = \mu_i \boldsymbol{p} \quad (M \times 1)$$

- For each good we have:

$$\frac{\partial u_i(\boldsymbol{x}_i)}{\partial x_{i,j}} = \mu_i p_j$$

  - The MRS for any two goods must be equal to the ratio of prices
  - Any two agents hold the same MRS since they face the same prices.

Definition

- **Competitive Equilibrium** is an allocation $\boldsymbol{x}^*$ and a price vector $\boldsymbol{p}^*$ such that:

  1. The allocation $\boldsymbol{x}_i^*$ solves agent i's problem given $\boldsymbol{x}^*$, for all i's.
  2. Market clears:

$$\sum_{i=1}^{n} x_{i,j}^* \leq \sum_{i=1}^{n} w_{i,j} \ \forall \ j$$

Definition

- An allocation $\boldsymbol{x}$ is **Pareto optimal** if it is feasible and there is no other feasible allocation $\tilde{\boldsymbol{x}}$ such that $u_i(\tilde{\boldsymbol{x}}_i') \geq u_i(\boldsymbol{x}_i)$ for all $i \in \{1, ..., N\}$, and $u_j(\tilde{\boldsymbol{x}}_j) > u_j(\boldsymbol{x}_j)$ for at least one $j \in \{1, ..., N\}$.
- **First Welfare Theorem** Every competitive allocation is Pareto optimal.
- Sketch of the proof:
  1. Assume $\tilde{\boldsymbol{x}}$ is preferable by at least one agent j and feasible.
  2. This allocation for agent j was out of his budget set with prices p.
  3. All other agents i cannot be consuming less and be as well off.
  4. Markets cannot clear $\Rightarrow$ allocation not feasible.

Social Planner's Problem

- Next, we would like to know whether every Pareto optimal allocation can be sustained by a competitive equilibrium.

- The set of Pareto optimal allocation can be characterized by the solution to the following planner's problem:

$$\max_{\boldsymbol{x}} \sum_{i=1}^{n} \alpha_i u_i(\boldsymbol{x}_i) \text{ with } \sum_{i}^{n} \alpha_i = 1$$

$$\text{s.t. } \sum_{i=1}^{n} \boldsymbol{w}_i = \sum_{i=1}^{n} \boldsymbol{x}_i$$

with α_i representing the weights of the different agents in the planner's objective.

Social Planner's Problem

- The solutions to the planner's problem is characterized by:

$$\alpha_i Du_i(\boldsymbol{x}_i) = \boldsymbol{\pi}$$
$$\sum_{i=1}^{n} \boldsymbol{w}_i = \sum_{i=1}^{n} \boldsymbol{x}_i^*$$

- The competitive allocation instead was characterized by:

$$Du_i(\boldsymbol{x}_i) = \mu_i \boldsymbol{p}$$
$$\boldsymbol{p}'(\boldsymbol{w}_i - \boldsymbol{x}_i) = 0$$
$$\sum_{i=1}^{n} \boldsymbol{w}_i = \sum_{i=1}^{n} \boldsymbol{x}_i$$

- Therefore if $\alpha_i = 1/\mu_i$ and $\boldsymbol{p} = \boldsymbol{\pi}$, the social planner and the competitive equilibrium coincide.

Social Planner's Problem

- Then, whether a Pareto optimal allocation can be decentralized boils down to whether at prices $\boldsymbol{\pi}$, the allocation $\boldsymbol{x}$ is feasible for each consumer.

- In order the allocation to be affordable to every agent, the planner has to redistribute income across agents:

$$\tau_i(\boldsymbol{\alpha}) = \boldsymbol{\pi}'(\boldsymbol{x}_i - \boldsymbol{w})$$

- Note that such redistribution comes at zero cost:

$$\sum_{i=1}^{n} \tau_i(\boldsymbol{\alpha}) = 0$$

Social Planner's Problem

- **Second Welfare Theorem** Every Pareto optimal allocation can be decentralized as a competitive equilibrium with transfers, i.e. given Pareto optimal allocation $\boldsymbol{x}$, we can find a price vector $\boldsymbol{p}$ and transfers $\boldsymbol{\tau}_i$ such that given the initial endowments and transfers, $\boldsymbol{x}$ is a competitive equilibrium.

Exchange Economy with Infinitively-Lived Agents

- In our static exchange economy agents live for a single period.
- In this section we will analyze model economies where they live forever.
- Time discrete, infinite, finite number of agents N, only one consumption good.
- The consumption good is not storable.
- Agents have deterministic endowments $w^i = \{w_t^i\}_{t=0}^{\infty}$

Exchange Economy with Infinitively-Lived Agents

- Let c_t^i be consumption of agent i at time t, and let $c^i = \{c_t^i\}_{t=0}^{\infty}$ be a consumption sequence.
- Agents preferences are given by,

$$U(c^i) = \sum_{t=0}^{\infty} \beta^t u_i(c_t^i),$$

where β is the discount factor.

- $U(c^i)$ is time separable.

Exchange Economy with Infinitively-Lived Agents

Market Structures

- We are going to study two system of markets:
  1. *Arrow-Debreu* structure with complete markets all trade takes place at time 0.
  2. Sequential trading structure with one period securities.
- These two structures will entail different assets and timing of trades but have identical consumption allocations.

Arrow-Debreu Markets

- There is a market at time 0 where agents can buy and sell goods of different time periods.
- There is a price for every period's good.
- We assume that all contracts that are agreed at time 0 are honored.
- The consumer therefore faces a single budget constraint:

$$\sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t w_t^i$$

- We call this market arrangement, Arrow-Debreu markets.
- We normalize $p_0 = 1$ (goods in period 1 are the *numeraire*)

Arrow-Debreu Equilibrium

- **Definition**: sequence of allocation $c^i = \{c_t^i\}_{t=0}^{\infty}$ for each i, and a sequence of prices $p = \{p_t\}_{t=0}^{\infty}$ such that:

1. Given p, c^i solves the agent i's maximization problem for each i:

$$\max_{c^i} \sum_{t=0}^{\infty} \beta^t u_i(c_t^i),$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t c_t^i \leq \sum_{t=0}^{\infty} p_t w_t^i$$

2. Markets clear for each t:

$$\sum_{i=1}^{n} c_t^i = \sum_{i=1}^{n} w_t^i$$

Arrow-Debreu Equilibrium

- The equilibrium allocations are characterized by:

1. Consumer's FOCs:

$$\beta^t \frac{\partial u_i(c_t^i)}{\partial c_t^i} = \mu^i p_t, \text{ for each } i \text{ and each } t$$

2. Individual's budget constraints

$$\sum_{t=0}^{\infty} p_t c_t^i = \sum_{t=0}^{\infty} p_t w_t^i$$

3. Aggregate resource constraint:

$$\sum_{i=1}^{n} c_t^i = \sum_{i=1}^{n} w_t^i$$

Arrow-Debreu Equilibrium

Intertemporal optimization

- From FOCs:

$$\frac{\beta^t \frac{\partial u_i(c_t^i)}{\partial c_t^i}}{\beta^{t+1} \frac{\partial u_i(c_{t+1}^i)}{\partial c_{t+1}^i}} = \frac{p_t}{p_{t+1}}$$

Intertemporal optimization conditions:

$$\frac{\partial u_i(c_t^i)}{\partial c_t^i} = \beta \frac{p_t}{p_{t+1}} \frac{\partial u_i(c_{t+1}^i)}{\partial c_{t+1}^i} \tag{1}$$

- The consumer allocates her resources optimally such that the marginal cost of reducing time-t consumption today equals the marginal benefit of increasing time-t+1 consumption tomorrow taking into account the discount factor and price dynamics.

Arrow-Debreu Equilibrium

- From FOCs:

$$\frac{\frac{\partial u_i(c_t^i)}{\partial c_t^i}}{\frac{\partial u_j(c_t^j)}{\partial c_t^j}} = \frac{\mu^i}{\mu^j}$$

- Therefore the ratio of marginal utilities across two agents is constant across time.

Arrow-Debreu Equilibrium

Example: Aggregate Time-Invariant Endowment

- Imagine that $\sum_{i=1}^{n} w_{it} = W$ constant through time.
- Then the aggregate ressource constraint can be written as:

$$\sum_{i=1}^{N} (u_c^i)^{-1}\left(\frac{\lambda_i}{\lambda_j} u_c^j(c_t^j)\right) = W$$

- As W is invariant, c_{jt} must be invariant too and therefore equation 1 becomes :

$$\begin{aligned} p_{t+1} &= \beta p_t \\ p_t &= \beta^t p_0 \\ p_t &= \beta^t \text{ w.l.o.g.} \end{aligned}$$

- Prices completely offset individuals impatience to induce them to maintain a constant consumption level.

Arrow-Debreu Equilibrium

Pareto Optimality of the Equilibrium

- **Proposition**: Any Arrow-Debreu equilibrium is Pareto optimal.
- Sketch of the proof:
  - Assume, it is not pareto optimal; there exists another feasible allocation $\tilde{c}$ such that

$$u(\tilde{c}^i) \geq u(c^i) \ \forall\, i$$
$$u(\tilde{c}^j) > u(c^j) \text{ for at least one } j$$

  - This implies that

$$\sum_{t=0}^{\infty} p_t \tilde{c}_t^j > \sum_{t=0}^{\infty} p_t c_t^j$$

  - Given that other individuals are on their budget set, adding across individuals and time:

$$\sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} \tilde{c}_t^i > \sum_{t=0}^{\infty} p_t \sum_{i=1}^{N} c_t^i$$

Pareto Optimal Allocation

- As before, we can characterize characterize the set of Pareto optimal allocations as solutions to the following planner's problem:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{i=0}^{n} \alpha_i \beta^t u_i(c_t^i)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} c_t^i = \sum_{i=1}^{n} w_t^i, \text{ for all t}$$

Pareto Optimal Allocation

- The solution to this problem is characterized by the following FOCs:

$$\alpha_i \beta^t \frac{\partial u_i(c_t^i)}{\partial c_t^i} = \pi_t, \text{ for all } i \text{ and } t,$$

where π_t is the Lagrange multiplier on the time-t constraint.

- Given α, allocations that solves the planner's problem are Pareto optimal.

Pareto Optimal Allocation

- In order to decentralize the Pareto optimal allocation, we use Lagrange multiplier as prices and transfer resources among consumers according to:

$$\tau_i(\alpha) = \sum_{t=0}^{\infty} \pi_t(\alpha)[c_t^i(\alpha) - w_t^i],$$

  where $c_t^i(\alpha)$ is the pareto optimal allocation of goods.

- We can use this framework to compute the Arrow-Debreu equilibrium by finding α^*, such that $\tau_i(\alpha^*) = 0$ for all i
  - The $\pi_t(\alpha^*)$ are the Arrow-Debreu prices and allocation c_t^i are the Arrow-Debreu allocations.

Sequential Equilibrium

Setup

- Our previous analysis was built on Arrow-Debreu markets where all trade takes place at time-0 market.
- Suppose now that trades takes place in *spot markets* that open every period.
- Hence, at time t; agents only trade time-t goods in a spot market.
- If agents can only trade time-t good at time t; and there are no credit arrangements, then this economy would look like a sequence of static exchange economies.

Sequential Equilibrium

Setup

- With spot markets we need a credit mechanism that will allow agents to move their resources between periods.
- Therefore, we will assume that there is a one period credit market that works as follows:
    - Each period, agents can borrow or lend in this one period credit market.
    - Let r_t be the interest rate on time-t borrowing/lending.

Sequential Equilibrium

Individual Problem

- Given a sequence of prices $\{r_t\}_{t=0}^{\infty}$, the agent i's problem can be written as:

$$\max_{\{c_t^i, l_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u_i(c_t^i)$$

$$\begin{aligned} \text{s.t.} \quad c_0^i + l_1^i &= w_0^i \\ c_1^i + l_2^i &= w_1^i + (1 + r_1) l_1^i \\ &\dots \\ c_t^i + l_{t+1}^i &= w_1^i + (1 + r_t) l_t^i \end{aligned}$$

Sequential Equilibrium

No-Ponzi Condition

- How can we make sure that agents don't borrow more than what they can honor?
- We are interested in specifying a borrowing limit that prevents Ponzi schemes, yet is high enough so that household are never constrained in the amount they can borrow.
- We need to impose an extra condition:

In t=0: $l_1^i = w_0^i - c_0^i$

In t=1: $l_2^i = w_2^i + (1 + r_1)w_0^i - c_1^i - (1 + r_1)c_0$

$\vdots$

In t: $$l_{t+1}^i = w_t^i + \sum_{s=0}^{t-1} \prod_{j=s+1}^{t} (1 + r_j)w_s^i - c_t^i - \sum_{s=0}^{t-1} \prod_{j=s+1}^{t} (1 + r_j)c_s$$

Sequential Equilibrium

No-Ponzi Condition

- Dividing both sides by $\prod_{j=1}^{t}(1+r_j)$:

$$\frac{l_{t+1}^i}{\prod_{j=1}^{t}(1+r_j)} = \sum_{s=1}^{t} \frac{w_s}{\prod_{j=1}^{s}(1+r_j)} + w_0 - \sum_{s=1}^{t} \frac{c_s}{\prod_{j=1}^{s}(1+r_j)} - c_0$$

- Which is simply the time-0 present value of agent's resources minus consumption.

- We need to impose a condition on it such that agents don't run a game where they keep borrowing and never pay back:

$$\lim_{t\to\infty} \frac{l_{t+1}^i}{\prod_{s=1}^{t}(1+r_s)} \geq 0$$

Sequential Equilibrium

No-Ponzi Condition

- The weakest possible debt limit would be to impose *the natural debt limit*:
    - It has to be feasible for the consumer to repay her debt at every time t.

$$l_{t+1}^{i} \geq -\sum_{s=t+1}^{\infty} \frac{w_s}{\prod_{j=t+1}^{s}(1+r_j)} + w_t$$

    - At every time t the value of her debt cannot exceed the discounted value of present and future endowments.

Sequential Equilibrium

FOCs

- From FOCs we get:

$$\beta^t \frac{\partial u_i(c_t^i)}{\partial c_t^i} = \lambda_t$$

$$\lambda_t = (1 + r_{t+1})\lambda_{t+1}$$

- Combining them,

$$\frac{\partial u_i(c_t^i)}{\partial c_t^i} = (1 + r_{t+1})\beta \frac{\partial u_i(c_{t+1}^i)}{\partial c_{t+1}^i}$$

Sequential Equilibrium

- **Definition** A sequential market equilibrium is a sequence of allocations $c^i = \{c_t^i\}_{t=0}^{\infty}$ and a sequence of lending/borrowing decisions $l^i = \{l_t^i\}_{t=0}^{\infty}$ for each i, and sequence of prices $r = \{r_t\}_{t=0}^{\infty}$ such that

  1. Given r, c^i and l^i solves agent's maximization problem
  2. Markets clear.

$$\sum_{i=1}^{n} w_t^i = \sum_{i=1}^{n} c_t^i \text{ for all } t$$

$$\sum_{i=1}^{n} l_t^i = 0 \text{ for all } t$$

Sequential Equilibrium

- **Proposition** *If* $\{c_t, p_t\}_{t=0}^{\infty}$ *is a competitive Arrow-Debreu equilibrium allocation, then letting:*

$$r_{t+1} = \frac{p_t}{p_{t+1}} - 1$$

$\{c_t, r_t\}_{t=0}^{\infty}$ *is a competitive equilibrium with sequential markets.*

- Sketch of the proof: If $r_{t+1} = \frac{p_t}{p_{t+1}} - 1$, c_t satisfies FOCs, markets clear and the no-ponzi condition is satisfied.

Stochastic Endowments

- So far we have analyzed economies where everything was certain.
- However, uncertainty is an important element in many economic activities.
- We are going to extend the previous analysis to a stochastic environment.

Stochastic Endowments

Setup

- Time discrete, infinite, N agents, one consumption good.
- Endowments depend on the history of states in the economy (s^t) which is uncertain: $w^i(s^t)$
- We will assume that state of the economy at a given time t (s_t) can take values from a given finite set S.

Arrow-Debreu Market

Setup

- We assume there is a time-0 Arrow-Debreu market where agents can buy and sell goods of different histories ($s^t = \{s_1, \ldots, s_t\}$).
  - Agents at time 0 choose a contingent plan where they decide her consumption for every date and every possible realization of the history.

$$c^i = \{c_t^i(s^t)\}_{t=0}^{\infty}$$

Arrow-Debreu Market

Agent Problem

- Agents maximize

$$U(c^i) = \max_{c^i} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t|s_0) u(c_t^i(s^t))$$

$$\text{s.t} \quad \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) w_t^i(s^t)$$

Arrow-Debreu Equilibrium

- **Definition**: An Arrow-Debreu equilibrium in this economy is a sequence of consumption plans c^i for each i, and a sequence of history dependent prices p such that given s_0,

  1. Given p, c^i solves agent's i maximization problem.
  2. Market clears

$$\sum_{i=1}^{n} c_t^i(s^t) \leq \sum_{i=1}^{n} w_t^i(s^t), \text{ for each } t \text{ and } s^t$$

Arrow-Debreu Equilibrium

FOCs

- By FOCs we get:

$$\beta^t \pi(s^t|s_0) \frac{\partial u(c_t^i(s^t))}{\partial c_t^i(s^t)} = \lambda p_t(s^t)$$

- Therefore the intertemporal FOC becomes:

$$\frac{\partial u(c_t^i(s^t))}{\partial c_t^i(s^t)} = \beta \frac{p_t(s^t)}{p_{t+1}(s^{t+1})} \pi(s^{t+1}|s^t) \frac{\partial u(c_{t+1}^i(s^{t+1}))}{\partial c_{t+1}^i(s^{t+1})}$$

Pareto Optimal Allocations

- As in the case without uncertainty, we can can characterize the set of Pareto optimal allocations as solutions to the following planner's problem:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{s^t} \sum_{i=0}^{n} \alpha_i \pi(s^t|s_0) \beta^t u_i(c_t^i)$$

$$\text{s.t.} \quad \sum_{i=1}^{n} c_t^i(s^t) = \sum_{i=1}^{n} w_t^i(s^t), \text{ for all t}$$

- Then, we could compute the competitive equilibrium by finding the set of α's such that the transfer function that you would need to sustain this equilibrium is 0 for all individuals.

Pareto Optimal Allocations

Perfect Insurance

- Note also that at time-t, history s^t consumption of any two agents is related by:

$$\frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} = \frac{\alpha_j}{\alpha_i}$$

- **Definition**: An allocation has perfect consumption insurance if the ratio of marginal utilities between two agents is constant across time (independent of the state of the world).

Pareto Optimal Allocations

Irrelevance of History

- From previous equation,

$$c_t^i(s^t) = u'^{-1}\left(\frac{\alpha_j}{\alpha_i}u'(c_t^j(s^t))\right)$$

- Summing across individuals and using aggregate resources constraint:

$$\sum_{i=1}^{I} w_t^i(s^t) = \sum_{i=1}^{I} u'^{-1}\left(\frac{\alpha_j}{\alpha_i}u'(c_t^j(s^t))\right)$$

which is one equation on one unknown $c_t^j(s^t)$

- The Pareto optimal allocations $\{c_t^i\}_{t=0}^{\infty}$ only depends on the aggregate state of the economy and not on the whole history.

Pareto Optimal Allocations

Irrelevance of History

- Assume $u(c) = \dfrac{c^{1-\sigma} - 1}{1-\sigma}$, then, we have:

$$c_t^i(s^t) = c_t^j(s^t)\left(\frac{\alpha_i}{\alpha_j}\right)^{1/\sigma}$$

- Given the feasibility constraint:

$$\sum_{i=1}^{I} c_t^j(s^t)\left(\frac{\alpha_i}{\alpha_j}\right)^{1/\sigma} = c_t^j(s^t)\left(\frac{1}{\alpha_j}\right)^{1/\sigma}\sum_{i=1}^{I}\alpha_i^{1/\sigma} = W(s^t)$$

which allows us to find

$$c_t^j(s^t) = \frac{\alpha_j^{1/\sigma}}{\sum_{i=1}^{I}\alpha_i^{1/\sigma}} W_t(s^t)$$

Agent j consumes a constant fraction of total endowment in every period.

Pareto Optimal Allocations

Irrelevance of History

- We can write the last expression in logs as:

$$\log c_t^j(s^t) = \log \theta_j + \log W_t(s^t)$$

or in first-differences, we could estimate using CEX data:

$$\Delta \log c_t^j(s^t) = \alpha_1 \Delta \log W_t(s^t) + \alpha_2 \Delta \log w_t^j(s^t) + \epsilon_{j,t}$$

- We get $\alpha_2 > 0$: excess sensitivity of consumption

Sequential Markets

Setup

- Suppose now that trade takes place sequentially in spot markets each period.
- Agents can buy and sell one period contingent claims or **Arrow securities** each period.
  - Securities that pay 1 unit of good at time $t+1$ for a particular realization of s_{t+1} tomorrow.
  - Let $Q(s_{t+1}, s^t)$ be the price of such contract at time t.
  - Let $a^i_{t+1}(s_{t+1}, s^t)$ be the purchase of agent i of such contract.
- Period t budget constraint is given by:

$$c^i(s^t) + \sum_{s_{t+1}} a^i_{t+1}(s^t, s_{t+1}) Q(s^t, s_{t+1}) = w^i_t(s^t) + a_t(s^t)$$

- Note that although the agent buys a portfolio of Arrow securities at time t, at $t+1$ only one of these securities will deliver returns.

Sequential Equilibrium

No-Ponzi Condition

- With a sequential market structure we again need to put a debt limit to rule out Ponzi schemes.
- A natural debt limit $A_t^i(s^t)$ for an agent can be calculated as

$$p_t(s^t)A_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) w_t^i(s^\tau),$$

$$\text{Debt limit: } -A_t^i(s^t) \leq a_t(s^t)$$

which means that the current value of your future endowments cannot be larger than the value of your debt using Arrow-Debreu prices.

Sequential Equilibrium

- **Definition**: A sequential market equilibrium in this economy is prices for Arrow securities $Q(s^t, s_{t+1})$ for all t and for all s^t, allocations $c_t^i(s^t)$ and $a_{t+1}^i(s^t, s_{t+1})$ for all agents, all t and all s^t such that

 1. For each i, given $Q(s^t, s_{t+1}), c_t^i(s^t)$ and $a_{t+1}^i(s_{t+1}, s^t)$ solve

$$\max_{c_t^i(s^t), a_{t+1}^i(s_{t+1}, s^t)} \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi(s^t | s_0) u(c^i(s^t))$$

$$\text{s.t.} \quad c_t^i(s^t) + \sum_{s_{t+1}} a_{t+1}^i(s^t, s_{t+1}) Q(s^t, s_{t+1}) = w_t^i(s^t) + a_t(s^t)$$

$$a_{t+1}^i(s^t, s_{t+1}) \geq -A_{t+1}^i(s^{t+1})$$

 2. Markets clear:

Agg. ressource constraint: $\sum_{i=1}^{n} w_t^i(s^t) = \sum_{i=1}^{n} c_t^i(s^t)$ for all s^t

Securities are in zero net supply: $\sum_{i}^{n} a_{t+1}^i(s^t, s_{t+1}) = 0$ for all s^t and s_{t+1}

Sequential Equilibrium

FOCs

- By FOCs we get:

$$Q(s^t, s_{t+1})u'(c_t^i(s^t)) = \beta\pi(s_{t+1}|s_t)u'(c_{t+1}^i(s^{t+1}))$$

from which we can see that if we let:

$$Q(s^t, s_{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)}$$

the allocations under the Arrow-Debreu and Sequential market structure coincide as the natural debt limit will not bind, the FOCs hold, and the aggregate ressource countraint holds.