Life-Cycle Models with Heterogeneous Agents

Jesús Bueren

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Introduction

Life cycle is a very important dimension for many questions:

- Accounting for the wealth distribution.
  Castañeda, Díaz-Giménez, and Ríos-Rull (2009)
- Social security programs transfer resources from workers to retirees.
  Fuster, İmrohoroğlu, İmrohoroğlu (2007)
- Tax reforms
  Conesa, Kitao and Krueger (2009)
- Human capital accumulation and endogenous earnings inequality has a clear life-cycle component.
  Ben Porath (1967); Hugget, Ventura and Yaron (2011)
- Portfolio choice.
  Cocco, Gomes, and Maenhout (2005)

Huggett (1996)

- Extension of Diamond (1965) OLG model.
  - Multi-period.
  - Lifetime uncertainty.
  - Income uncertainty.
- It can also be seen as Aiyagari (1994) w/ life cycle.
- First serious attempt at accounting for the wealth distribution
- Results:
  - It matches the large Gini index of the US wealth distribution.
  - It does so through a counterfactually large share of people in zero wealth and too little concentration at the top.

Huggett (1996)

Setup

- Life-cycle dimension:
  - The average labor income changes with age.
  - Households retire at age J_R.
  - The probability of surviving to the next period is age-dependent
    In period J the probability of dying is 1

- Stationary age distribution:
  - Each period a continuum of households of size $\bar{N}_t$ are born.
  - New cohorts may grow in size at a constant rate $\bar{N}_{t+1} = (1+n)\bar{N}_t$.
  - The survival probabilities are time-independent.

- Stationary economy:
  - No aggregate uncertainty.
  - Wealth and income distribution of first cohort identical across time.

- Standard production side.

Households

Setup

- Labor market income $e(z,j)w$
  - w is the market wage rate common to all agents.
  - $e(z,j)$ is the productivity of agents at j with idiosyncratic productivity z.
    (after retirement, age $j = J_R$, it will be zero)
  - $z \in \boldsymbol{Z} \equiv \{z_1, z_2, \ldots, z_M\}$ and follows a Markov process $\Gamma_{z,z'}$.

- There is a PAYG social security system,pays $b_j = b > 0$ for $j \geq J_R$.

- Agents can save and borrow through a risk free asset a:
  - to smooth out the life-cycle earnings profile.
  - to self-insure against earnings uncertainty.
  - to self-insure against excessive longevity risk.

  There is a lower bound $\underline{a}$ on the holdings of this asset.

  More generally, we establish $a \in \boldsymbol{a} \equiv [\underline{a}, \bar{a}]$

Households

Decision Problem

- Households have preferences over consumption at different points in time.

- At birth, expected utility is given by:

$$E\left[\sum_{j=1}^{J}\beta^{j-1}\left(\prod_{i=1}^{j}s_i\right)u(c_j)\right]$$

  where s_i are conditional survival probabilities.

- The budget constraints they face are of the type:

$$c_j + a_{j+1} = a_j R + (1-\theta)e(j,z)w + T + b_j$$

  T denotes accidental bequests, θ is the social security payroll tax and b_j the social security transfer.

- The feasibility and terminal constraints:
  $c_j \geq 0,\ a_j \geq \underline{a},\ a_1, z_1$ given, and $a_{j+1} \geq 0$ if $j = J$

A Note on Social Security

- It is important to introduce a public PAYG social security as in data:

  1. It helps generate the right incentives for retirement savings:
     - PAYG social security substitutes private savings
       (PAYG $\Rightarrow$ Lower aggregate capital in steady state)
     - Public pensions are paid out as life annuities
       (insurance against excessive longevity risk $\Rightarrow$ lower savings incentives)

  2. It helps produce a sizeable share of asset-poor households.

- In this formulation, the author does not link pensions to contributions. This implies that there is:
  - Lower uncertainty in the model economy.
  - Low incentives to save for income-poor households.
  - High incentives to save for income-rich households.
    (The model generates inequality through a wrong channel)

Household Problem

Recursive Problem

- The HH problem in recursive form:

$$v_j(a,z) = \max_{a',c} \left\{ u(c) + s_j \beta \sum_{z'} \Gamma_{z',z} v_{j+1}(a',z') \right\}$$
$$\text{s.t. } c + a' = aR + (1-\theta)e(j,z)w + b_j + T$$
$$a' \geq \underline{a} \text{ and } c \geq 0$$

- The standard Euler equation:

$$u_c(aR + (1-\theta)e(j,z)w + b_j + T - a')$$
$$= s_j \beta R \sum_{z'} \Gamma_{z,z'} u_c(a'R + (1-\theta)e(j+1,z')w + b_{j+1} + T - a'')$$

- We are looking for policy function $g_j^a(a,z)$ and $g_j^c(a,z)$

Solving the Household Problem

Backwards Induction

- Analogous to value function iteration.
- In the life-cycle problem, the Bellman equation is not stationary: $v_{j+1}(a, z)$ is a different function than $v_j(a, z)$.
- Hence, we do not look for a fixed point exploiting the Contraction Mapping Theorem.
- Instead, we solve by backwards induction:
  - Period J is the last one. Hence we know that:

    $$g_J^a(a, z) = 0 \text{ and } g_J^c(a, z) = aR + (1 - \theta)e(J, z)w + b_J + T$$

  - Hence the value at J:

    $$v_j(a, z) = u(g_J^c(a, z))$$

  - From here on, we can solve backwards for every period j because we know v_{j+1}

Solving the Household Problem

Backwards Induction

In period j do as follows:

- Solve:

$$v_j = \max_{a',c}\{u(c) + s_j\beta \sum_{z'} \Gamma_{z,z'} v_{j+1}(a', z')\}$$

$$\text{s.t. } c + a' = aR + (1-\theta)e(j,z)w + b + T$$

$$a' \geq \underline{a} \text{ and } c \geq 0$$

where $v_{j+1}(a', z')$ is known from $j+1$ period solution

- Obtain $g_j^a(a,z)$ and $g_j^c(a,z)$.
- Obtain the value function:

$$v_j(a,z) = u(g_j^c(a,z)) + s_j\beta \sum_{z'} \Gamma_{z,z'} v_{j+1}(a', z')$$

- Move on and solve for period $j-1$.

Solving the Household Problem

Using the Euler Equation

- The same idea of backwards induction can be applied in the Euler equation when looking for the policy function.
- Let's discretize the space a of our endogenous state variable into a dimension-I real-valued vector $\tilde{a} = \{\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_I\}$.
- Let's define $J \quad M \times I$ matrices $\tilde{g}_j^a$, where M is the number of elements of the earnings space Z and I is the number of elements of $\tilde{a}$.
- Every element $\{m, i\}$ of the matrix $\tilde{g}_j^a$ states the choice a' for an individual of type $\{z_m, \tilde{a}_i\}$ at age j.
- Our approximation $\hat{g}_j^a$ to the true policy function g_j^a is constructed by linear interpolation of $\tilde{g}_j^a$

Solving the Household Problem

Using the Euler Equation

- Let's define
$$d_j(w,z) = (1-\theta)e(j,z)w + b_j + T$$
as the non-financial income for individual of age j with shock z
- Then, we can write the Euler equation as,

$$0 = u_c[d_j(w,z) + Ra - \hat{g}^a_j(z,a;\tilde{g}^a_j)] -$$
$$s_j\beta R\sum_{z'}\Gamma(z,z')u_c[d_{j+1}(w,z') + R\hat{g}^a_j(z,a;\tilde{g}^a_j) - \hat{g}^a_{j+1}(z',\hat{g}^a_j(z,a;\tilde{g}^a_j);\tilde{g}^a_{j+1})]$$

- Knowing the matrix $\tilde{g}^a_{j+1}$ the Euler equation delivers a matrix $\tilde{g}^a_j$:
  - At J, agents are constrained so they are not on their Euler equation: we know that $\tilde{g}^a_J = 0$
  - Then at $j = J-1$, knowing $\tilde{g}^a_J$ we can solve for $\tilde{g}^a_{J-1}$
  - Iterating backwards, we can solve by all $\tilde{g}^a_j$ j with knowledge of $\tilde{g}^a_{j+1}$

Solving the Household Problem

Using the Euler Equation

1. Obtain $\tilde{g}^a_J$. Set $j = J - 1$.

2. Obtain $\tilde{g}^a_j$ given $\tilde{g}^a_{j+1}$
   - For every pair $\{z_l, \tilde{a}_i\} \in \boldsymbol{Z} \times \tilde{a}$ we look for $a' \in \boldsymbol{a}$ that solves the following non-linear equation:

$$\begin{aligned} 0 = & u_c[d_j(w, z) + R\tilde{a}_i - a'] \\ & - s_j \beta R \sum_{z'} \Gamma(z, z') u_c[d_{j+1}(w, z') + Ra' - \hat{g}^a_{j+1}(z', a'; \tilde{g}^a_{j+1})] \end{aligned}$$

   - Notice that this is just one equation in one unknown, a', which is not restricted to lie on the grid $\tilde{a}$. We set $\tilde{g}^a_j(m, i) = a'$.
   - Doing this for all possible values $z_m \in \boldsymbol{Z}$ and all possible values $\tilde{a}_i \in \tilde{a}$ we obtain the whole matrix $\tilde{g}^a_j$.

3. Set $j = j - 1$ and go back to step 2.

Firm's Problem

- The firm's problem is very standard.
- We assume Cobb-Douglas production function.
- Firm's maximize:

$$\max_{L,K} K^{\alpha} L^{1-\alpha} - (r+\delta)K - wL$$

- FOC:

$$\begin{aligned} \alpha K^{\alpha-1} L^{1-\alpha} &= r+\delta \\ (1-\alpha) K^{\alpha} L^{-\alpha} &= w \end{aligned}$$

- The wage is a function of the interest rate and L which is given because of inelastic labor supply.

Steady State Equilibrium

Definition

A steady state equilibrium for this economy is:

- a set of functions $\{v_j, g_j^a, g_j^c\}_{j=1}^J$
- a pair of aggregate allocations K and L (in per capita terms)
- an amount of transfers T (in per capita terms)
- a series of probability measures $\{\mu_j\}_{j=1}^J$
- a series of transition functions $\{Q_j\}_{j=1}^J$
- a pair of prices $\{w, r\}$
- a pair of social security parameters $\{\theta, b\}$

such that

Steady State Equilibrium

Definition

- Households solve their optimization problem.
  That is to say, given a pair of prices $\{w, r\}$ and social security parameters $\{\theta, b\}$, the functions $\{v_j, g_j^a, g_j^c\}_{j=1}^{J}$ solve the hh problem.

- Firms solve their optimization problem
  Factor prices are given by the first order conditions of the firm:

$$R = 1 + F_K(K/L) - \delta \text{ and } w = F_L(K/L)$$

- Labor market clears

$$\sum_{j=1}^{J_R-1} \psi_j \int_{\boldsymbol{Z} \times a} e(z, j) d\mu_j = L$$

- Capital market clears

$$\sum_{j=1}^{J} \psi_j \int_{\boldsymbol{Z} \times a} g_j^a(z, a) d\mu_j = K' = K$$

Steady State Equilibrium

Definition

- The social security administration is in balance

$$\theta w L = b \sum_{j=J_R}^{J} \psi_j$$

- Accidental bequests are given back as transfers,

$$\sum_{j=1}^{J} \psi_j (1 - s_j) \int_{\boldsymbol{Z} \times a} R g_j^a(z, a) d\mu_j = T' = T$$

- The measures of households at each age is given by,

$$\mu_{j+1}(B) = \int_{\boldsymbol{Z} \times a} Q_j(b, B) d\mu_j \text{ and } \mu_1, \text{ given}$$

- The transition functions Q_j arise from the optimal behavior of households and the markov chain Γ.

- Goods market clears:

$$F(K, L) + (1 - \delta)K = \sum_{j=1}^{J} \psi_j \int_{\boldsymbol{Z} \times a} (g_j^a(z, a) + g_j^c(z, a)) d\mu_j$$

Calibration

- Demographics
  Life tables to obtain s_j , average population growth to obtain n
- Income process
  Estimate from panel data: deterministic age component and residual
- Social security b and θ
  Match average replacement rate in the data and budget balance
- Technology parameters δ, α
  I/Y and capital share
- Preferences parameters σ, β
  Standard values off the shelves
- Borrowing limit, $\underline{a}$
- Initial conditions: μ_1
  Zero wealth and earnings dispersion of young households.

Calibration

Social Security

- The social security payroll tax θ is calibrated analytically.
  - Let's call ω the average replacement ratio in the data.
  - Then, we want the model to satisfy

$$\omega = \frac{b}{wL} \sum_{j=1}^{J_R-1} \psi_j \text{ and } \theta w L = b \sum_{j=J_R}^{J} \psi_j$$

  - Both expressions together give

$$\theta = \omega \frac{\sum_{j=J_R}^{J} \psi_j}{\sum_{j=1}^{J_R-1} \psi_j}$$

    - So, with ω from the data we recover analytically the payroll tax θ
- The pension b is calibrated together with the equilibrium algorithm

Steady State Equilibrium

How to find it?

1. Algorithm starts at iteration k with a guess on r_k
2. Obtain prices K_k^d, w_k and the social security parameter b_k

$$R_k = 1 + F_K(K_k^d/L) - \delta \text{ and } w_k = F_L(K_k/L) \text{ and } \theta w_k L = b_k \sum_{j=J_R}^{J} \psi_j$$

3. Iterate to find accidental bequests
   - 3.1 Guess transfers T_k^g
   - 3.2 Solve hh problem with T_k^g
   - 3.3 Aggregate and compute accidental bequests T_k^{g+1}
   - 3.4 If they are equal go on. Otherwise set $T_k^{g+1} = T_k^g$ and come back to (3.2)
4. Aggregate household savings $K_k^s = \sum_{j=1}^{J} \psi_j \int_{\boldsymbol{Z}\times\boldsymbol{a}} g_j^a(z,a) d\mu_j$
5. If $|K_k^s - K_k^d| < \epsilon$, stop. Otherwise set $R_{k+1} = 1 + F_K(K_k^s/L) - \delta$ and back to 2

Aggregating

In Theory

- We keep track of the population in the economy by means of
  - ψ_j, the fraction of individuals with age j (exogenous).
  - $\mu_j(B)$, the probability measure that tells us the density of individuals of age j in any subset $B \subset \boldsymbol{Z} \times \boldsymbol{a}$ of the state space.
  - The law of motion for μ_j is given by,

$$\mu_{j+1}(B) = \int_{\boldsymbol{Z} \times a} Q_j(b, B) d\mu_j$$

  - Hence, note that there are J distributions μ_j , one for every age group.
  - Notice that we need to give an initial condition μ_1, which describes the joint distribution of assets and labor earnings of every cohort that enters the labor market.

Aggregating

In Practice: Monte-Carlo Simulation

- Take an initial finite sample $\hat{\mu}_1$

  (This should be a calibration sample)

- At any period j, take $\hat{\mu}_j$, use the $\hat{g}_j^a$, the $\Gamma_{z',z}$, and a random number generator to compute $\hat{\mu}_{j+1}$.

- In this manner, you end up with J distributions $\hat{\mu}_j$.

- Then, the ψ_j can be computed deterministically

  (there is no need to kill anybody)

  - Compute the cross-sectional age distribution at period t:

$$\tilde{\psi}_{t,j+1} = s_j \tilde{\psi}_{t,j} (1+n)^{-1} \text{ and } \tilde{\psi}_{t,1} = \bar{N}_t$$

  - And then normalize by population size such that the ψ_j sum up to one:

$$\psi_{j=a} = \frac{\tilde{\psi}_{j=a}}{\sum_{j=1}^{J} \tilde{\psi}_j}$$