Chapter II: A Model of Production

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The Economy as a Model

- In the previous chapter we saw that economists rely on mathematical models to interpret how the economy works.
- In macroeconomics we focus on economies as systems in which many markets interact simultaneously.
- A model is said to be in general equilibrium when both prices and quantities are determined within the model.
- Today we introduce our first general equilibrium framework, built around three key markets:
  - Labor market
  - Capital market
  - Goods market

Production Function

- A production function tells us how a firm turns inputs into output:

$$Y = F(K, L)$$

- Imagine we run an ice cream factory. How much ice cream can we make with:
  - L workers
  - K machines
- What the symbols mean:
  - Y is our total ice cream production
  - L is labor (workers or hours worked)
  - K is capital (machines that do the hard work)
- Labor plus capital are the basic ingredients of production.

Properties of Production Functions

- What makes a production function realistic?
- If we add more workers while keeping machines fixed:
  - Output increases (more hands help)
  - But each new worker adds a little less than the previous one
- The same happens with machines: more machines help, but not forever.

Properties of Production Functions

- We describe these ideas using marginal products.
- Positive marginal products:

$$\frac{\partial F}{\partial L} \geq 0, \qquad \frac{\partial F}{\partial K} \geq 0$$

- Diminishing marginal products:

$$\frac{\partial^2 F}{\partial L^2} \leq 0, \qquad \frac{\partial^2 F}{\partial K^2} \leq 0$$

- Translation: more inputs always help, but each extra unit helps a little less.

Cobb–Douglas Production Function

- A very popular production function in macro is the Cobb–Douglas:

$$Y = AK^a L^{1-a}$$

- A measures productivity: how good the economy is at turning inputs into output.
- The marginal products are easy to compute:

$$\frac{\partial Y}{\partial L} = (1-a)AK^a L^{-a} \geq 0, \qquad \frac{\partial Y}{\partial K} = aAK^{a-1}L^{1-a} \geq 0$$

- And both marginal products get smaller as we increase that input:

$$\frac{\partial^2 Y}{\partial L^2} \leq 0, \qquad \frac{\partial^2 Y}{\partial K^2} \leq 0$$

- The two inputs also help each other:

$$\frac{\partial^2 Y}{\partial K \partial L} = a(1-a)AK^{a-1}L^{-a} \geq 0$$

  Adding machinery raises how much extra output you get from hiring a worker

Returns to Scale

- Now suppose our technology is:

$$Y = AK^{a}L^{1-a}$$

- What happens if we double all inputs, both K and L?

$$F(2K,2L) = A(2K)^{a}(2L)^{1-a} = 2AK^{a}L^{1-a} = 2F(K,L)$$

- Doubling every input gives exactly double the output.
- We call this constant returns to scale.

Why Constant Returns to Scale?

- For Cobb–Douglas, returns to scale depend on the exponents:
- If the exponents sum to one ($a + (1 - a) = 1$): constant returns to scale.
- If they sum to more than one: increasing returns.
- If they sum to less than one: decreasing returns.

Behavior of Firms

- How should we model the behavior of firms in economics?
- Usual assumptions:
  - Individuals maximize utility
  - Firms maximize profits
- But firms are tricky... what exactly is a firm?
  - Shareholders put up money and "own" the firm
  - Managers are hired by shareholders
  - Employees are hired by managers
  - Creditors lend money to the firm
- Lots of potential conflicts of interest! (Shareholders vs managers, managers vs employees...)

Firm's Problem

- Firms choose how many workers to hire and machines to rent to maximize profits:

$$\max_{K,L} F(K,L) - rK - wL$$

- Profits = production minus cost of inputs
- We set the price of output to 1 (numeraire)
- r = rental price of capital, w = wage of workers
- Firms take r and w as given because markets are competitive (simplifying assumption — left-wing critique possible)

Firm's Problem: Solution

- How do we solve the profit-maximization problem?

$$\max_{K,L} F(K,L) - rK - wL$$

- Step 1: Choose optimal capital K
  Maximize profits with respect to K, treating L as fixed
- Step 2: Choose optimal labor L
  Maximize profits with respect to L, treating K as fixed

Maximization in Two Variables

- We use a simple but powerful math result:

$$\max_{x,y} f(x,y)$$

- The solution satisfies the two first-order conditions:

$$\frac{\partial f(x^*,y^*)}{\partial x} = 0, \qquad \frac{\partial f(x^*,y^*)}{\partial y} = 0$$

- This is exactly what we will do for the firm's profit problem.

Firm's Problem with Cobb–Douglas

- Profit function:

$$\Pi(K,L) = F(K,L) - rK - wL$$

- Plug in Cobb–Douglas production:

$$\Pi(K,L) = AK^{a}L^{1-a} - rK - wL$$

- Goal: choose K and L to maximize profits.

Optimal Capital Choice

- Differentiate profit with respect to K, holding L constant:

$$\frac{\partial \Pi}{\partial K} = aAK^{a-1}L^{1-a} - r$$

- Set derivative equal to zero (first-order condition):

$$aAK^{a-1}L^{1-a} - r = 0$$

- Solve for K in terms of L and parameters:

$$aAK^{a-1}L^{1-a} = r$$

Optimal Labor Choice

- Differentiate profit with respect to L, holding K constant:

$$\frac{\partial \Pi}{\partial L} = (1-a)AK^{a}L^{-a} - w$$

- Set derivative equal to zero (first-order condition):

$$(1-a)AK^{a}L^{-a} - w = 0$$

- Solve for L in terms of K and parameters:

$$(1-a)AK^{a}L^{-a} = w$$

Firm's Profit Maximization Conditions

- The firm maximizes profits if:

$$aAK^{a-1}L^{1-a} = r, \qquad (1-a)AK^{a}L^{-a} = w$$

- Let's interpret it.
- These are the first-order conditions: optimal rules for hiring capital and labor.

Optimal Choice of Capital

- First-order condition for capital:

$$aAK^{a-1}L^{1-a} = r$$

- RHS: price of capital (rental rate)
- LHS: marginal product of capital (MP_K)

$$\frac{\partial Y}{\partial K} = aAK^{a-1}L^{1-a}$$

- Intuition: Rent capital up to the point where its extra output (the marginal product of capital) equals the price of capital: r.

Optimal Choice of Labor

- First-order condition for labor:

$$(1-a)AK^{a}L^{-a} = w$$

- RHS: price of labor (wage)
- LHS: marginal product of labor (MP_L)

$$\frac{\partial Y}{\partial L} = (1-a)AK^{a}L^{-a}$$

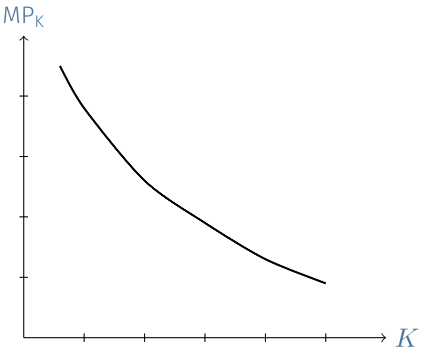
- Intuition: Hire labor up to the point where its extra output (the marginal product of labor) equals the price of labor w.

Firm Behavior

- The marginal product of capital is:

$$\text{MP}_\text{K} = aAK^{a-1}L^{1-a}$$

- For fixed L, A, a, this gives us a curve in (K, MP_K) space:
- Why is it downward sloping?
  - The marginal product of capital falls when K rises.

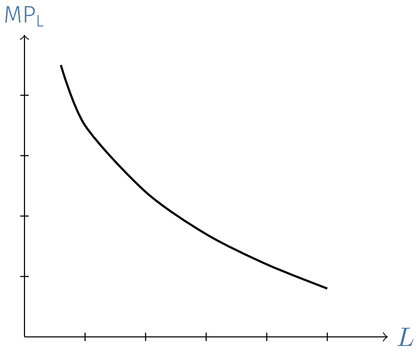


Firm Behavior

- The marginal product of labor is:

$$\mathrm{MP_L} = (1-a)AK^{a}L^{-a}$$

- For fixed K, A, a, this gives a curve in $(L, \mathrm{MP_L})$ space:
  - All points (L, MPL) that satisfy the condition.
- Why is it downward sloping?
  - The marginal product of labor falls as L increases.



Power of Competition

- We assume the labor market is perfectly competitive.
- Under perfect competition, workers are paid their marginal product:

$$w = \text{MP}_L$$

- Why can't firms pay workers less?
  - If a firm underpays, another firm can profitably hire those workers away.
  - Competition among firms forces wages up to marginal product.

Power of Competition: A Caution

- Perfect competition is a strong assumption and often unrealistic.
- We sometimes adopt it for analytical convenience.
- The danger: we may forget we made the assumption and start believing markets are always competitive or efficient.
- Daniel Kahneman called this "theory-induced blindness": failing to see deviations from a model once we grow used to the model.

Capital and Labor Supply

- Simplifying assumption: labor and capital supplied inelastically
- We assume the total amount of capital is fixed. We call this fixed amount $K^s = \overline{K}$.
- To keep things simple, we also assume the total amount of labor is also fixed. We call this fixed amount $L^s = \overline{L}$.

Model Summary

- The model consists of five equations:
- Capital demand: $aAK^{a-1}L^{1-a} = r$
- Labor demand: $(1-a)AK^{a}L^{-a} = w$
- Capital supply: $K = \overline{K}$
- Labor supply: $L = \overline{L}$
- Production function: $Y = AK^{a}L^{1-a}$
- The five endogenous variables are: K, L, r, w, and Y.

An Equilibrium

- In economics, the solution to a model is called an equilibrium.
- Equilibrium describes what happens when all markets clear, that is, supply equals demand in each market.
- In our model, we have five equations with five unknowns.
- Solving the model means finding the values of the endogenous variables (K, L, r, w, Y) in terms of the exogenous variables ($\overline{K}$, $\overline{L}$, A, a).
- This involves rewriting the system so that all endogenous variables are on one side of the equations and only parameters and exogenous variables on the other side.

Equilibrium Values

- In our simple model, the equilibrium values are:
- Capital: $K = \overline{K}$
- Labor: $L = \overline{L}$
- Rental rate of capital: $r = aA\overline{K}^{a-1}\overline{L}^{1-a}$
- Wage: $w = (1-a)A\overline{K}^{a}\overline{L}^{-a}$
- Output: $Y = A\overline{K}^{a}\overline{L}^{1-a}$
- These values satisfy all five equations of the model and clear labor, capital and goods markets.

Labor Share in Our Model

- Labor compensation is wL. The labor share of output is:

$$\text{Labor share} = \frac{wL}{Y}$$

- From the labor demand curve:

$$w = (1-a)AK^{a}L^{-a} = (1-a)\frac{Y}{L}$$

- Multiply both sides by L to get total labor compensation:

$$wL = (1-a)Y$$

- Therefore, labor receives a constant fraction $1-a$ of total output in this model.

Why Cobb-Douglas

- Cobb-Douglas is easy to work with mathematically.
- More importantly, it implies that the share of output going to each factor is constant over time.
- Empirically, labor and capital shares have been surprisingly stable since World War II: $\alpha = 1/3$

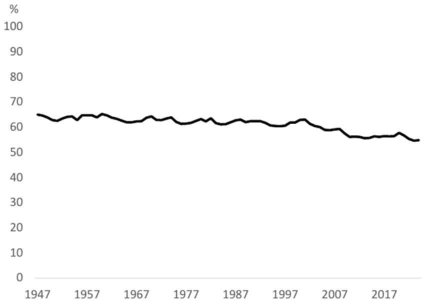


Figure 2: Labor Share in U.S. Nonfarm Business Sector

Development Accounting

- We now make an important leap: we apply the production function of our model to aggregate economies across the world.
- Output per person in the model, y, is matched to **GDP per capita**.
- Capital, k, is measured as the economy's stock of housing, factories, machines, and equipment, divided by population.
- The production function implies:

$$y = Ak^{1/3}.$$

- For now, we make a strong simplifying assumption:

$$A = 1 \quad \text{for all countries,}$$

so that income differences depend only on capital per person:

$$y = k^{1/3}.$$

- To ease comparisons, we normalize all variables so that

$$y_{\text{US}} = k_{\text{US}} = 1.$$

Predictions of the Model

Consider Portugal as an example.

- Capital per person in Portugal and Spain is close to or even slightly above the U.S. level.

According to the model:

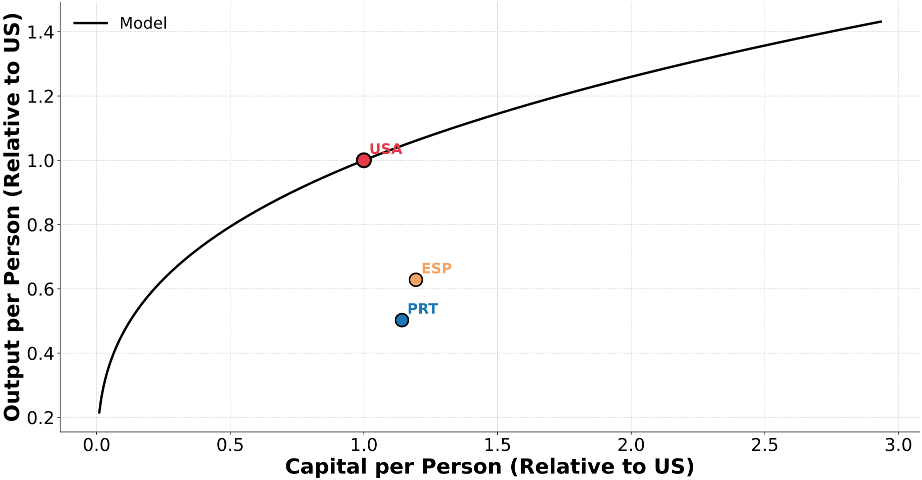
$$y = k^{1/3},$$

so if $k_{\text{PRT}} \approx 1$, the model predicts:

$$y_{\text{PRT}} \approx 1.$$

Prediction: Portugal should be about as rich as the United States.

Production Model: Model vs Data



Source: Penn World Table 11.0 (Feenstra, Inklaar, and Timmer). Year: 2023.

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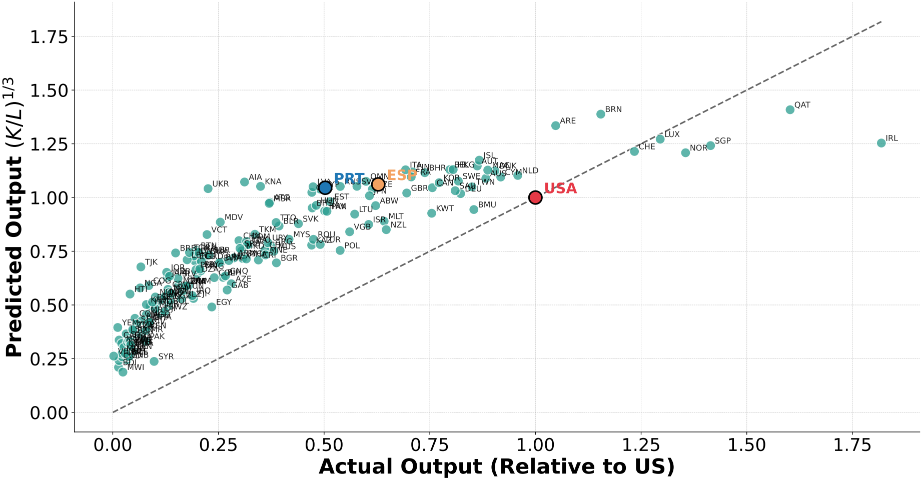

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Model vs Data

- In general, the model correctly suggests that countries will be rich or poor according to how much capital per person they have.
- The magnitudes are really wrong.
- One way to reconcile model with data, would be to allow the productivity parameter A to become country specific
- A measures how productive countries are at using their factor inputs (in this case K and L) to produce output.
- Economist assume their model is correct to estimate the value of A: for example, what would be the value of A that would rationalize that in reality Portugal produces much less that the US?
- The estimated A is referred as total factor productivity.

Model vs Data

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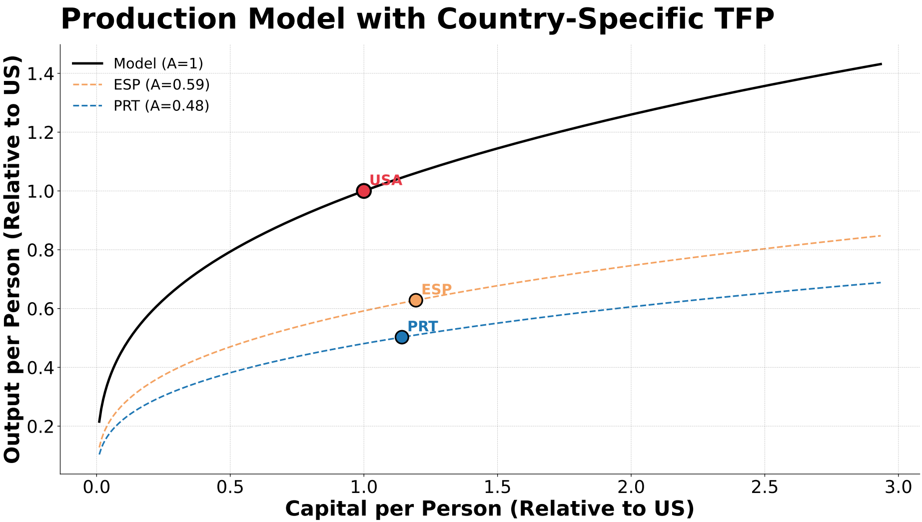
- However, the magnitudes are very wrong.
- One way to reconcile the model with the data is to allow the productivity parameter A to be country-specific.
- A measures how productive countries are at using their factor inputs (in this case, K and L) to produce output.
- Economists typically assume the model is correct and use it to infer the value of A. For example: what value of A would rationalize why Portugal produces much less output per capita than the U.S.?
- The inferred value of A is referred to as total factor productivity (TFP).



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Model vs Data

- Poor countries have 70 times lower GDP per capita than the US
- If poor countries had the same capital per person but the TFP of the US, they would be around 5 times richer
- Therefore differences, if poor countries has the US TFP they would be 14 times richer.
- What drives differences in TFP?
  - Education
  - Technoloy
  - Institutions
  - Misallocation

Worries about Labor-Saving Technology

- People have worried since the Industrial Revolution that machines will take jobs.
- Despite large amounts of labor-saving technology, labor share has remained roughly constant.
- This is very different from what happened with horses, which were replaced by machines and effectively disappeared from production.
- The question today: will new technologies like AI and robots change labor share in the future?

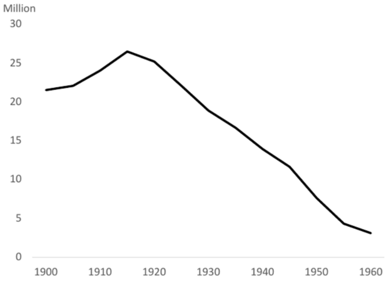

Figure 3: Number of Horses and Mules in the United States, 1900-1960

- One reason labor share might fall is better or cheaper machines.
- In a Cobb-Douglas production function:

$$Y = A(zK)^a L^{1-a}$$

where z measures machine quality.

- The rental rate of capital is:

$$r = aAK^{a-1}L^{1-a}z^a = a\frac{Y}{K}$$

- Total payments to capital:

$$rK = aY$$

- Labor share remains $1-a$, independent of z. Cobb-Douglas cannot capture a falling labor share due to better machines.

Is AI Going to Steal Our Jobs?

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- Labor share remains $1-a$, independent of z. Cobb-Douglas cannot capture a falling labor share due to better machines.
- With a different production function (e.g., CES with elasticity $>$ 1), labor share falls if machines improve or become cheaper.